Sequential learning of SAR image time series for monitoring Earth deformation

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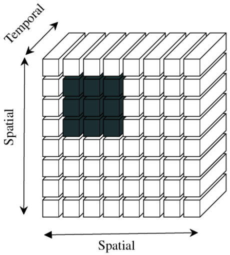
Mathematical modeling of multi-temporal SAR data



For a given pixel x_i with p temporal observations:

$$x_i = [x_i^1, x_i^2, ..., x_i^p]^T$$

An homogeneous neighbourhood (similar backscatterring and statistical properties): $\Omega = \{x_i\}_{i=1}^n$

Second order statistic: $E[x^q(x^l)^H] = \gamma_{q,l}\sigma_q\sigma_l e^{j(\theta_q-\theta_l)}$
in matrix form

$$\mathbb{E}[\mathrm{xx}^H] \triangleq \Sigma = \mathrm{ediag}(\boldsymbol{w_\theta}) \underbrace{((\boldsymbol{\sigma\sigma}^\top) \circ \boldsymbol{\Gamma})}_{\Psi} \mathrm{ediag}(\boldsymbol{w_\theta})^H$$

with $w_\theta = [e^{j\theta_1}, \ldots, e^{j\theta_p}]$

$$\Sigma(\boldsymbol{\Psi}, \theta) = \mathrm{mod}(\Sigma) \odot \arg(\Sigma) \triangleq \boldsymbol{\Psi} \odot (\mathrm{w}_{\boldsymbol{\theta}}\mathrm{w}_{\boldsymbol{\theta}}^H)$$

Σ complex covariance matrix — interferogram

$\boldsymbol{\Psi}$ module of the covariance matrix

$\boldsymbol{\Gamma}$ coherence matrix

$\boldsymbol{w_\theta}$ vector of phase exponential

State of the art: Phase Linking

- Objective: Estimate $p-1$ phase differences from p SAR images $\longrightarrow$ $\boldsymbol{w_\theta}$
- Main idea: minimize the negative log-likelihood

$$\begin{array}{ll} \underset{\boldsymbol{\Psi}, \boldsymbol{w_\theta}}{\min} & \mathcal{L}(\boldsymbol{\Sigma}(\boldsymbol{\Psi}, \boldsymbol{\theta})) \\ \text{subject to} & \boldsymbol{\Psi} \text{ real and symmetric} \\ & \theta_0 = 0 \end{array}$$

Assuming that $\boldsymbol{\Psi}$ is known, the method is equivalent to optimizing the following problem:

$$\begin{array}{ll} \underset{\boldsymbol{w_\theta}}{\min} & \boldsymbol{w_\theta}^H (\boldsymbol{\Psi}^{-1} \circ \boldsymbol{S}) \boldsymbol{w_\theta} \\ \text{subject to} & \theta_1 = 0 \end{array}$$

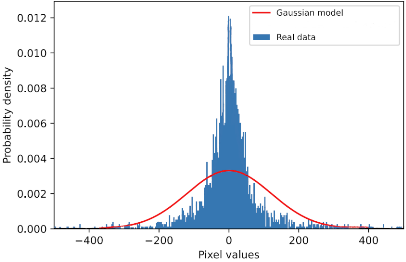
with $\boldsymbol{S}$ sample covariance matrix

In reality, $\boldsymbol{\Psi}$ is unknown

- [1] proposed to use a plug-in estimator: $\boldsymbol{\Psi} = |\boldsymbol{S}| \longrightarrow$ **not optimal**
- [2] proposed to estimate $\boldsymbol{\Psi}$ jointly with $\boldsymbol{w_\theta}$

[1] A. Guarnieri et al. (2008) "On the exploitation of target statistics for SAR interferometry applications".
[2] P.V.H. Vu et al. (2022) "A new phase linking algorithm for multi-temporal InSAR based on the Maximum Likelihood Estimator".

State of the art: Phase Linking



Empirical distribution of a Sentinel-1 image (July 3 2019) over the Mexico City.

- Gaussian distribution

$$\{\boldsymbol{x}^i\}_{i=1}^n \quad \text{an i.i.d set,} \longrightarrow \boldsymbol{x} \sim \mathcal{CN}(0, \boldsymbol{\Sigma})$$

$$\mathcal{L}(\boldsymbol{x}; \boldsymbol{\Sigma}) = -\log\left(\prod_{i=1}^n f(\boldsymbol{x}^i, \boldsymbol{\Sigma})\right)$$
$$\propto n\log(|\boldsymbol{\Sigma}|) + n\,\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{S})$$

- Scaled Gaussian distribution

$$\{\boldsymbol{x}^i\}_{i=1}^n, \quad \text{an i.i.d set, } \boldsymbol{x}^i \sim \mathcal{CN}(0, \tau_i\boldsymbol{\Sigma})$$

$$\mathcal{L}(\boldsymbol{x}; \boldsymbol{\Sigma}, \{\tau_i\}_{i=1}^n) = n\log|\boldsymbol{\Sigma}| + n\mathrm{Tr}\{\boldsymbol{\Sigma}^{-1}\mathrm{S}\}$$
$$+ \sum_{i=1}^n p\log\tau_i + \text{const}$$

$$\boxed{\rightarrow \text{closed-form solutions in both cases}}$$

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[1] A. Guarnieri et al. (2008) "On the exploitation of target statistics for SAR interferometry applications".
[2] P.V.H. Vu et al. (2022) "A new phase linking algorithm for multi-temporal InSAR based on the Maximum Likelihood Estimator".

State of the art: covariance fitting

Minimizing a dissimilarty measure between an empirical covariance matrix and the theoretical covariance matrix.

$$\begin{array}{ll} \min\limits_{\mathrm{w}_{\boldsymbol{\theta}}} & f^d(\tilde{\Sigma}, \boldsymbol{\Psi} \odot \mathrm{w}_{\boldsymbol{\theta}} \mathrm{w}_{\boldsymbol{\theta}}^H) \\ \text{subject to} & \theta_1 = 0 \end{array}$$

Possible options

Matrix distance f^d:

- Kullback - Leibler divergence
- Frobenius norm (least square)

Plag-in (empirical) estimator $\tilde{\Sigma}$:

- Sample covariance
- M-estimator
- Phase-only matrix ($re^{j\phi} \to e^{j\phi}$)

Considering decorrelation mechanism

Regularization:

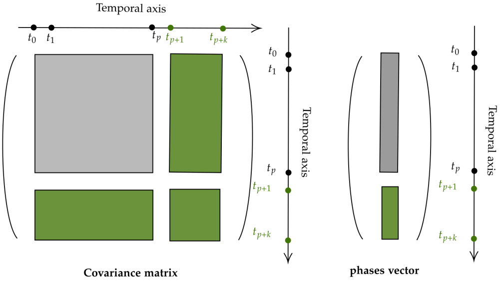
- Low rank
- Banding
- Shrinkage

$$\tilde{\Sigma}(\beta) = \beta \hat{\Sigma} + (1 - \beta) \frac{\mathrm{tr}(\hat{\Sigma})}{p} \mathrm{I}$$

with $\beta \in [0, 1]$.

[3] P.V.H. Vu et al. (2024) "Covariance Fitting Interferometric Phase Linking: Modular Framework and Optimization Algorithms"

When more images are coming ...



$$\forall i = 1, \ldots, n$$

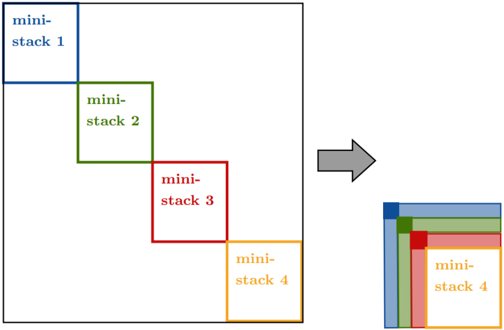
$$\tilde{x}^i = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_p^i \end{pmatrix}_{(p)} \rightarrow \tilde{x}^i = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_p^i \\ x_{p+1}^i \\ \vdots \\ x_{p+k}^i \end{pmatrix}_{(p+1)}$$

Integrating **new** SAR acquisitions

- **Estimation** of the increasing covariance matrix, **Re-Estimation** of the past covariance matrix
- **Estimation** of the new phases, **Re-Estimation** of the past phases

$\longrightarrow$ **Huge computation time**
$\longrightarrow$ **Need of sequential approaches**

State-of-the-art: Sequential estimator



Processing steps:

- Mini stack division: split the time series into small temporal blocks
- Phase linking: estimate phase values in each block
- Date compression: reduce the data to one virtual image for each block
- Datum connection: link all mini-stacks by a global phase linking

Potential drawbacks

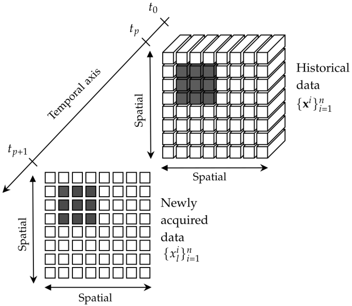
- Need time to get meaningful mini stack
- Based on Gaussian assumptions on SAR data
- Lack of robustness against various decorrelation mechanism

[4] H. Ansari et al. (2017) "Sequential Estimator: Toward efficient InSAR time series analysis".

Contributions

- Sequential Phase Linking based on Maximum Likelihood Estimation (S-MLE-PL)
- Sequential Covariance Fitting Interferometric Phase Linking (S-COFI-PL)
- SLiding interferometric Phase Linking (SL-IPL)

Sequential MLE Phase Linking



- Updated data

$$\tilde{\boldsymbol{x}}^i = [\underbrace{x_1^i, \ldots, x_p^i}_{\boldsymbol{x}^i}, x_l^i]^T \in \mathbb{C}^{l=p+1}$$

- Incremental update of the covariance

$$\tilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \boldsymbol{\Sigma} & w_{\theta_l}^* \operatorname{diag}(\boldsymbol{w_\theta})\boldsymbol{\gamma}^T \\ \boldsymbol{\gamma} \operatorname{diag}(\boldsymbol{w_\theta})^H w_{\theta_l} & \gamma_l \end{pmatrix}$$

- Incremental update of the log-likelihood (scaled Gaussian distribution) - closed form solution

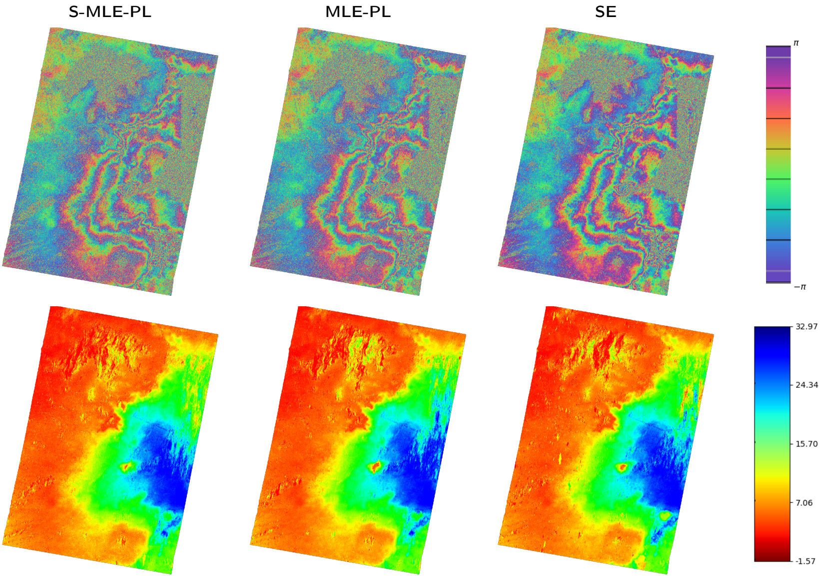
$$\begin{array}{cl} \underset{\boldsymbol{\gamma}, \gamma_l, w_{\theta_l}, \{\tau_i\}_{i=1}^n}{\min} & \mathcal{L}(\tilde{\boldsymbol{x}}^i; \boldsymbol{\gamma}, \gamma_l, w_{\theta_l}, \{\tau_i\}_{i=1}^n) \\ \text{subject to} & \theta_1 = 0, |w_{\theta_l}| = 1, \boldsymbol{\gamma}, \gamma_l \text{ real} \end{array}$$

Summary: w_θ is estimated with the past $\hat{w}_\theta$ and $\hat{\Sigma}$ as well as the new data

$\rightarrow$ conditional statistics of the new phase on the past.

D. El Hajjar et al. (2024) "Sequential Phase Linking : Progressive Integration of SAR Images for Operational Phase Estimation."
D. El Hajjar et al. (2025) "Robust Sequential Phase Estimation Using Multi-Temporal SAR Image Serie"

Sequential MLE Phase Linking



Mexico City, Sentinel-1, 14/04/2019 - 10/04/2020, time series of 20 images, SE with blocks of 5 images.

Sequential MLE Phase Linking

- **S-MLE-PL**: uses $p = 19$ historical images and one new image.
- **MLE-PL** [6]: uses all $l = 20$ available images.
- **SE** [4]: uses a temporal sliding window of size $s = 5$ images.

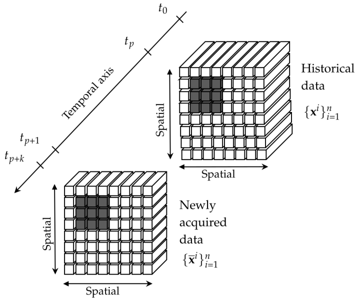
	RMSE (↓)	SSIM (↑)	Colinearity (↑) [3]	Computation time
MLE-PL [6]	Ref	Ref	0.61	38.4 min
S-MLE-PL	2.87	0.70	0.57	**4.72 min**
SE [4]	2.96	0.67	0.76	16.97 min

with a 95-core CPU running at 2.2 GHz and 125 G of RAM with calculations executed in parallel across the CPUs.

SSIM: Structural SIMilarity

[4] H. Ansari et al. (2017) "Sequential Estimator: Toward efficient InSAR time series analysis".
[5] B. Pinel-Puysségur et al. (2012) "Multi-link InSAR timeseries: Enhancement of a wrapped interferometric database".
[6] P.V.H. Vu et al. (2023) "Robust Phase Linking in InSAR".

Sequential Covariance Fitting Interferometric Phase Linking



- Updated data vector

$$\widetilde{\mathbf{x}}^i = [\underbrace{x_1^i, \ldots, x_p^i}_{\mathbf{x}^i}, \underbrace{x_{p+1}^i, \ldots, x_{p+k}^i}_{\overline{\mathbf{x}}^i}]^T \in \mathbb{C}^{p+k}$$

- Incremental update of the covariance

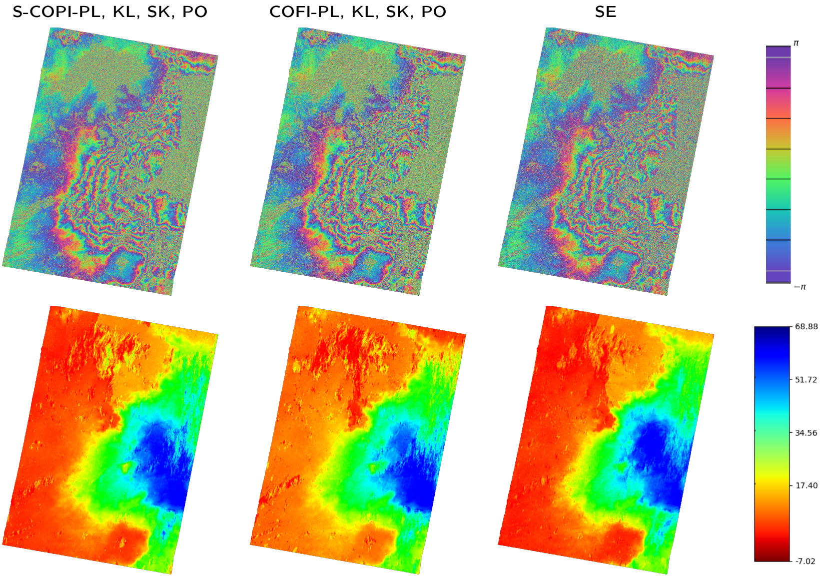
$$\widetilde{\Sigma} = \begin{pmatrix} \Sigma_p & (\Sigma_{pn})^H \\ \Sigma_{pn} & \Sigma_n \end{pmatrix}$$

- Optimization (Minimization - Majorization algorithm)

$$\begin{array}{ll} \underset{\overline{\mathbf{w}}_\theta}{\min} & f^{\mathrm{d}}_{\widetilde{\Sigma}}\left(\widetilde{\Sigma}, \widetilde{\Psi} \circ \widetilde{\mathbf{w}}_\theta \widetilde{\mathbf{w}}_\theta^H\right) \\ \text{subject to} & \theta_1 = 0 \\ & \overline{\mathbf{w}}_\theta \in \mathbb{T}_k \end{array}$$

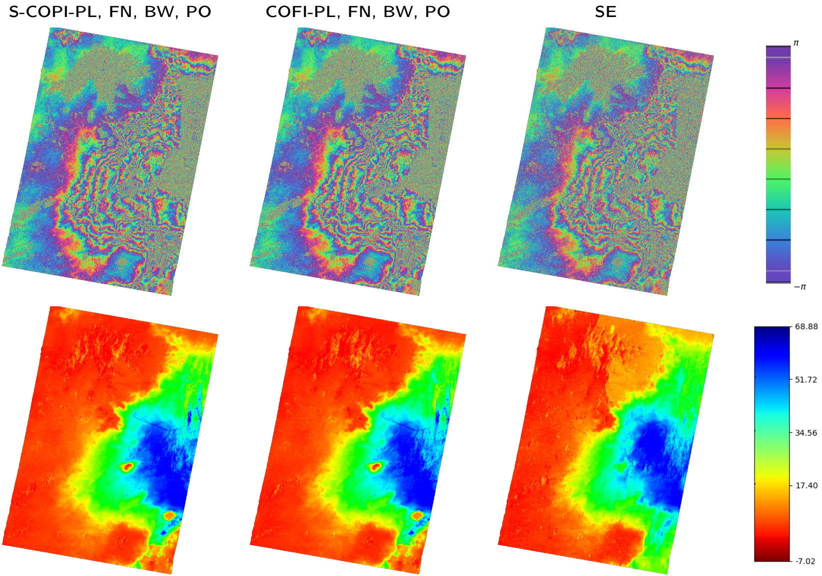
D. El Hajjar et al. (2025) "Sequential Covariance Fitting for InSAR Phase Linking".

Sequential Covariance Fitting Interferometric Phase Linking



Mexico City, Sentinel-1, 14/04/2019 - 06/12/2020. time series of 40 images, block of 5 images.

Sequential Covariance Fitting Interferometric Phase Linking



Mexico City, Sentinel-1, 14/04/2019 - 06/12/2020. time series of 40 images, block of 5 images.

Sequential Covariance Fitting Interferometric Phase Linking

- **S-COFI-PL**: uses $p = 35$ historical images and $k = 5$ new images.
- **COFI-PL** [7]: uses all $l = 40$ available images.
- **SE** [4]: uses a temporal sliding window of size $s = 5$ images.

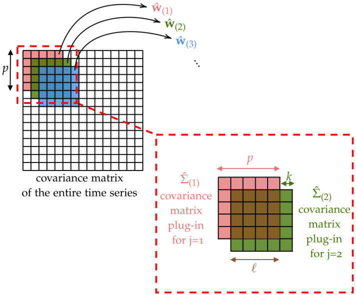
		RMSE (↓)	SSIM (↑)	Colinearity [5] (↑)	Computation time
KL	**S-COFI-PL**	7.54	0.62	**0.64**	**0.44**h
SK-PO	COFI-PL	7.73	0.55	0.60	0.52h
FN	**S-COFI-PL**	**2.07**	**0.89**	**0.89**	**0.37**h
BW-PO	COFI-PL	Ref	Ref	**0.89**	0.43h
	SE [4]	7.47	0.24	0.72	0.5h

with a 95-core CPU running at 2.2 GHz and 125 G of RAM with calculations executed in parallel across the CPUs.

KL : Kullback-Leibler divergence; FN : Frobenius Norm; BW : banding; SK : Shrinkage-to-identity; PO : Phase-Only; SSIM : Structural SIMilairy

Sliding interferometric Phase Linking

Temporal sliding window:



p: size of the temporal window

k: size of the stride or the new added images

$\ell = p - k$: size of the overlap

No need of storing all the historical data — memory and computational efficiency

- Covariance fitting formulation

$$\begin{aligned} \min_{\mathrm{w}_{(j)}} \quad & f^{FN}_{\hat{\boldsymbol{\Sigma}}_{(j)}}\left(\boldsymbol{\Sigma}_{(j)}, \boldsymbol{\Psi}_{(j)} \circ \mathrm{w}_{(j)}\mathrm{w}_{(j)}{}^{H}\right) \\ & + \lambda h(\mathrm{w}_{(j)}) \\ \text{subject to} \quad & \mathrm{w}_{(j)} \in \mathbb{T}_p \end{aligned}$$

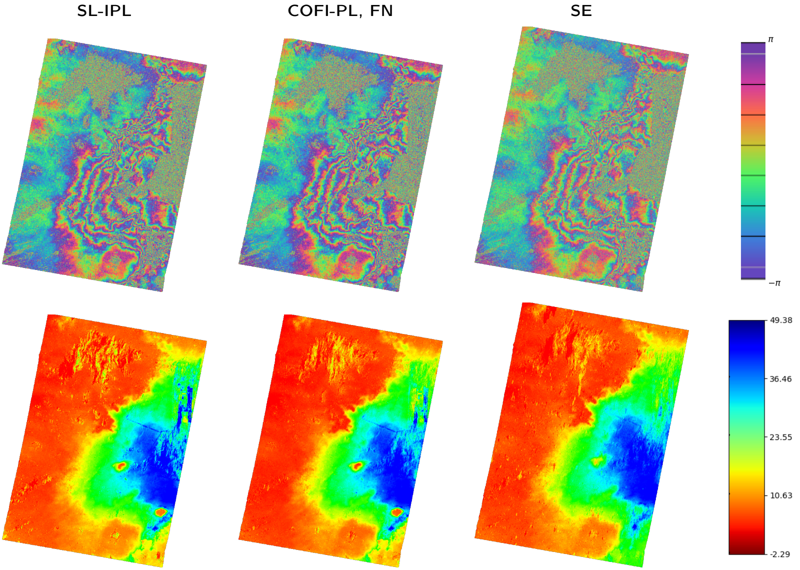
with a penalty term h that guarantees the phase estimates to stay close to the previous ones in the window overlap

$$h(\mathrm{w}_{(j)}) = \left\| \begin{pmatrix} \mathrm{w}_{\ell(j-1)} \\ 0 \end{pmatrix} - \mathrm{w}_{(j)} \right\|^2$$

- Optimization: Minimization - Majorization algorithm

D. El Hajjar et al. (2025) "Sliding IPL: An efficient approach for estimating the phases of large SAR image time series

Sliding interferometric Phase Linking



Mexico City, Sentinel-1, 14/04/2019 - 06/12/2020. time series of 30 images, block of 5 images, stride = 1.

Sliding interferometric Phase Linking

- **SI-IPL**: uses $p = 5$, $\ell = 4$, and $k = 1$.
- **COFI-PL** [7]: uses all $l = 30$ available images.
- **SE** [4]: uses a temporal sliding window of size $s = 5$ images.

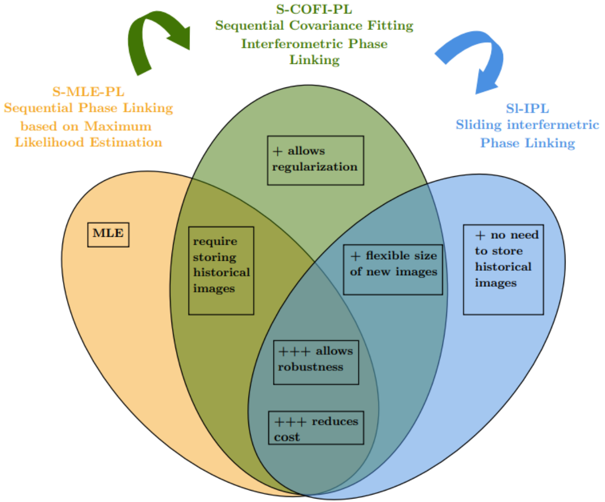
method	colinearity [5] (↑)	SSIM (↑)	Computation time
COFI-PL [7]	**0.91**	Ref	19.35 min
SI-IPL	**0.90**	**0.94**	**3.9** min
SE [4]	0.84	0.85	17.1 min

with a 95-core CPU running at 2.2 GHz and 125 G of RAM with calculations executed in parallel across the CPUs.

SSIM: Structural SIMilarity

[4] H. Ansari et al. (2017) "Sequential Estimator: Toward efficient InSAR time series analysis".
[5] B. Pinel-Puysségur et al. (2012) "Multi-link InSAR timeseries: Enhancement of a wrapped interferometric database".

Conclusions



Codes availability:
https://github.com/DanaElhajjar/S-G-MLE-PL
https://github.com/DanaElhajjar/S-COFI-PL
https://github.com/DanaElhajjar/S-SG-MLE-PL
https://github.com/DanaElhajjar/SI-IPL